term is negative. This does not automatically hold here. The g term also enters in the no ghost condition, $\alpha_K+(3/2)\alpha_B^2\geq 0$, in both α_B and the kineticity α_K [14]. Moreover, g and α_B also enter in $G_{\rm eff}$, so even in the limit where the density and pressure are dominated by the kinetic term the theory and its observational implications lie outside k-essence.

V. DEVIATING THE KINETIC STRUCTURE

Moving away from the natural case where all ratios are of order one, suppose XK_X/K is not of order one. If it is very small, this is of no concern since then simply $\rho_{\rm de}\approx -K$ and $w\approx -1$. However large XK_X/K requires special attention. This is because while it enters in $\rho_{\rm de}$, only K enters in $P_{\rm de}$. If $\rho_{\rm de}/K$ becomes large enough, then the α_B term in the pressure could dominate over the first term.

Consider a Dirac-Born-Infeld (DBI) kinetic term (see e.g. [16, 17]). Then

$$K(X) = -\sqrt{1-X/X_\infty} \tag{16}$$

$$n \equiv \frac{XK_X}{K}=\frac{-X/X_\infty}{2(1-X/X_\infty)}\ , \tag{17}$$

and $n\to-\infty$ as $X\to X_\infty$. As seen by Eq. (15) this would drive $w\to 0^-$, or at least to $\mathcal{O}(\alpha_B)$. Thus an interesting scenario for the future of dark energy is that it could end up scaling as matter.

VI. DEVIATING THE FIELD EVOLUTION

Changing $\ddot\phi/(3H\dot\phi)$ from order one also has interesting consequences. Consider $\ddot\phi\gg 3H\dot\phi$. If the cosmic expansion rate H is the only timescale then this cannot be achieved, so such a condition requires introduction of a new time scale.

Recall that in the standard Klein-Gordon equation one has the acceleration term $\ddot\phi$, the friction term $3H\dot\phi$, and the driving term from the steepness of the potential, $-dV/d\phi$. As clearly illustrated in [18] for the quintessence case, none of these terms dominate over the others (unlike in slow roll inflation). Here we have no potential, but the driving term arises from $-6g(\dot H+3H^2)$.

One needs an extra ingredient entering the driving term, an additional, shorter, time scale than the expansion time scale H^{-1}. One example is a mass scale M, e.g.

$$g(X)=\gamma(X)+M\ . \tag{18}$$

However this appears problematic as a large $\ddot\phi$, increasing at faster than the Hubble rate, will eventually lead to a similarly large $\dot\phi$ and hence a large $\alpha_B\sim\dot\phi g/H$. That will cause large deviations from GR, with $G_{\rm eff}$ possibly diverging and instability from likely negative sound speed of scalar perturbations, $c_s^2<0$ (see [15]). So this case appears problematic.

VII. DEVIATING BY SLOW ROLL

We can take the opposite limit of field evolution: $\ddot\phi=0$, known as slow roll in inflation. Recall that we absolutely cannot do this for generic quintessence when the dark energy does not fully dominate [18], i.e. as in our present or past universe where there is matter. Taking $\ddot\phi=0$ would imply $\phi\sim t$, i.e. the only clock is the field, hence the field must dominate the cosmic expansion.

Suppose the $\ddot\phi=0$ condition holds, at some time at least. The field equation (5) then has the interesting property that it determines the action function g in terms of K (though only in this limit),

$$g=\frac{-\dot\phi K_X}{3H(1-w_{\rm dom})}\ , \tag{19}$$

where we have used $\dot H=(-3/2)(1+w_{\rm dom})H^2$, and $w_{\rm dom}$ indicates the equation of state of the dominant energy density component[2].

Keeping $\alpha_B\ll 1$ will imply $w\to-1$ when dark energy dominates, as

$$\alpha_B\equiv\frac{2\dot\phi g}{H}=\frac{-4XK_X}{3H^2(1-w)}\approx\frac{4n}{(1-2n)(1-w)}\ . \tag{20}$$

Thus $\alpha_B\ll 1$ requires $|n|\ll 1$, and hence by Eq. (15) that $w\to-1$. This implies that after dark energy crosses the phantom divide to $w>-1$, in this specific $\ddot\phi=0$ future it must turn around and go back to a de Sitter limit.

VIII. NO KINETIC TERM

We know that the g (i.e. G_3) term must exist, otherwise we are left with k-essence, which cannot cross the phantom divide by itself. Let us consider whether we can do without the K term, having only the G_3 term in the action.

When $K=0$, the field equation gains an interesting structure,

$$0 = \ddot\phi\, 6H\dot\phi g_X+6g\left(\dot H+3H^2\right) \tag{21}$$

$$= 6a^{-3}\left[(a^3H)\dot g+g\left(a^3H\right)^{\cdot}\right]\ , \tag{22}$$

since $\dot g=g_X\dot X=g_X\dot\phi\ddot\phi$. The solution is

$$g\sim\left(a^3H\right)^{-1}\ . \tag{23}$$

[2] This solution is quite similar to the Branch A solution of [19] that can well temper, i.e. cancel, a high energy cosmological constant, when we have a de Sitter state, with $w_{\rm dom}=-1$, $H\to h=$const. However canceling Λ only at late times is not so useful.